

CONGRUENCES FOR ANDREWS' SMALLEST PARTS PARTITION FUNCTION AND NEW CONGRUENCES FOR DYSON'S RANK

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ABSTRACT. Let $\text{spt}(n)$ denote the total number of appearances of smallest parts in the partitions of n . Recently, Andrews showed how $\text{spt}(n)$ is related to the second rank moment, and proved some surprising Ramanujan-type congruences mod 5, 7 and 13. We prove a generalization of these congruences using known relations between rank and crank moments. We obtain explicit Ramanujan-type congruences for $\text{spt}(n) \bmod \ell$ for $\ell = 11, 17, 19, 29, 31$ and 37 . Recently, Bringmann and Ono proved that Dyson's rank function has infinitely many Ramanujan-type congruences. Their proof is non-constructive and utilizes the theory of weak Maass forms. We construct two explicit nontrivial examples mod 11 using elementary congruences between rank moments and half-integer weight Hecke eigenforms.

1. INTRODUCTION

Let $\text{spt}(n)$ denote the number of smallest parts in the partitions of n . Below is a list of the partitions of 4 with their corresponding number of smallest parts. We see that $\text{spt}(4) = 10$.

4	1
$3 + 1$	1
$2 + 2$	2
$2 + 1 + 1$	2
$1 + 1 + 1 + 1$	4

In a recent paper Andrews [3] showed that $\text{spt}(n)$ is related to the second rank moment. The rank of a partition [19] is the largest part minus the number of parts. The crank of a partition [4] is the largest part if the partition has no ones, otherwise it is the difference between the number of parts larger than the number of ones and the number of ones. We let $N(m, n)$ denote the number of partitions of n with rank m . For $n \neq 1$ we let $M(m, n)$ denote the number of partitions of n with crank m . For $n = 1$ we define

$$M(-1, 1) = 1, M(0, 1) = -1, M(1, 1) = 1, \quad \text{and otherwise } M(m, 1) = 0.$$

For k even we define

$$N_k(n) = \sum_m m^k N(m, n) \quad (k\text{th rank moment}),$$

$$M_k(n) = \sum_m m^k M(m, n) \quad (k\text{th crank moment}).$$

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Andrews proved that

$$(1.1) \quad \text{spt}(n) = n p(n) - \frac{1}{2} N_2(n),$$

where $p(n)$ is the number of partitions of n . Dyson [20] gave a combinatorial proof that

$$(1.2) \quad \frac{1}{2} M_2(n) = n p(n).$$

Hence we have

$$(1.3) \quad \text{spt}(n) = \frac{1}{2} (M_2(n) - N_2(n)).$$

We make the following

Conjecture 1.1.

$$M_k(n) > N_k(n),$$

for k even, $k \geq 2$ and $n \geq 1$.

The case $k = 2$ follows from (1.3). We have checked the conjecture for $k \leq 10$ and $n \leq 500$.

In [3], Andrews found some surprising congruences for $\text{spt}(n)$

$$(1.4) \quad \text{spt}(5n + 4) \equiv 0 \pmod{5},$$

$$(1.5) \quad \text{spt}(7n + 5) \equiv 0 \pmod{7},$$

$$(1.6) \quad \text{spt}(13n + 6) \equiv 0 \pmod{13}.$$

These congruences are reminiscent of Ramanujan's partition congruences

$$(1.7) \quad p(5n + 4) \equiv 0 \pmod{5},$$

$$(1.8) \quad p(7n + 5) \equiv 0 \pmod{7},$$

$$(1.9) \quad p(11n + 6) \equiv 0 \pmod{11}.$$

Andrews' proof of (1.4), (1.5) depends solely on (1.7), (1.8) and known relations for the rank mod 5 and 7. The proof of (1.6) is more difficult and depends on relations mod 13 for the rank due to O'Brien [24].

In Section 2 we prove the following generalization of (1.4)–(1.6): For $t = 5, 7$ or 13 ,

$$\text{spt}(n) \equiv -2(n + \frac{t^2-1}{24}) p(n) \pmod{t},$$

provided $1 - 24n$ is not a quadratic residue mod t . The proof uses known relations between rank and crank moments [8].

In Section 3 we prove a number of results on Hecke operators and congruences for modular forms that will be needed in later sections. In Section 4 we study $\text{spt}(n) \pmod{11}$. By using a known relation between rank moments, crank moments and the 23rd power of the eta-function we find that

$$\sum_{n=0}^{\infty} \text{spt}(11n + 6) q^n \equiv 4 \prod_{n=1}^{\infty} (1 - q^n)^{13} \pmod{11}.$$

By using the fact that $\eta(24\tau)^{13}$ is a Hecke eigenform we obtain the congruence

$$\text{spt}(11 \cdot 19^4 \cdot n + 22006) \equiv 0 \pmod{11}.$$

In Section 5 we show that the generating functions for the rank moments $N_{2k}(11n + 6) \pmod{11}$ can basically be written in terms of half-integral weight Hecke eigenforms. As a result we are able to find the following explicit congruences for the rank

$$(1.10) \quad N(r, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11},$$

$$(1.11) \quad N(r, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11},$$

for all $0 \leq r \leq 10$. Bringmann and Ono [15] conjectured and Bringmann [12] proved that for each prime $\ell > 3$ and $m, n \in \mathbb{N}$ there are infinitely many non-nested arithmetic progressions $An + B$ such that

$$(1.12) \quad N(r, \ell^m, An + B) \equiv 0 \pmod{\ell^u},$$

for all $0 \leq r \leq \ell^m - 1$. The congruences (1.10) and (1.11) represent the first nontrivial explicit examples of this result. The analogue of (1.12) for $N(r, t, An + B)$ when $(t, 2\ell) = 1$ had been proved earlier by Bringmann and Ono [15]. The crank analogue was proved by Mahlburg [23]. All these results generalize the analog for $p(n)$ proved by Ono [25]. It is also clear by (1.1) that $\text{spt}(n)$ can be written as an integer-linear combination of the rank functions $N(r, \ell^u, n) \pmod{\ell^u}$. Thus Bringmann's result (1.12) also implies that there are infinitely many non-nested arithmetic progressions $An + B$ such that

$$(1.13) \quad \text{spt}(An + B) \equiv 0 \pmod{\ell^u}.$$

In Section 6 we describe algorithms for computing congruences for the generating functions of $\text{spt}(\ell n + \beta_\ell) \pmod{\ell}$ when $24\beta_\ell \equiv 1 \pmod{\ell}$, and Ramanujan-type congruences for $\text{spt}(n)$. For certain small primes we find that an appropriate form for the generating function of $\text{spt}(\ell n + \beta_\ell)$ is congruent mod ℓ to a half-integer weight Hecke eigenform. This leads to explicit examples of (1.13) which we list below.

$$\begin{aligned} \text{spt}(11 \cdot 19^4 \cdot n + 22006) &\equiv 0 \pmod{11}, \\ \text{spt}(7^4 \cdot 17 \cdot n + 243) &\equiv 0 \pmod{17}, \\ \text{spt}(5^4 \cdot 19 \cdot n + 99) &\equiv 0 \pmod{19}, \\ \text{spt}(13^4 \cdot 29 \cdot n + 18583) &\equiv 0 \pmod{29}, \\ \text{spt}(29^4 \cdot 31 \cdot n + 409532) &\equiv 0 \pmod{31}, \\ \text{spt}(5^4 \cdot 37 \cdot n + 1349) &\equiv 0 \pmod{37}. \end{aligned}$$

These explicit congruences are reminiscent of congruences for the partition function $p(n)$ found earlier by Atkin [6], [7], and Weaver [32]. The connection with half-integer weight Hecke eigenforms is also analogous to what happens for the partition function. This was exploited by Guo and Ono [26], who showed for small ℓ a connection between ℓ -divisibility results for the orders of certain Tate-Shafarevich groups of certain Tate twists of Dirichlet motives, and partition congruences. Guo and Ono's results should extend to spt -congruences.

Recently, Folsom and Ono [21] have proved some amazing congruences for $\text{spt}(n) \pmod{2}$ and 3 . These congruences had been observed by the author and others. The methods of the present paper do not apply to Folsom and Ono's results. In the present paper all congruences have prime modulus ℓ where $\ell > 3$.

Recently, Bringmann [13] has shown how the generating function for the second rank moment gives rise to a weak Maass form of weight $3/2$. This leads to an asymptotic formula and congruences, which in turn also implies the result (1.13).

In the paper [14] we further explore the general problem of congruences mod ℓ for rank moments, Andrews [2] symmetrized rank moments and full rank functions for k -marked Durfee symbols for general ℓ . In Section 7 we will present a preview of some of these results and make some concluding remarks.

2. CONGRUENCES FOR $\text{spt}(n) \pmod{5, 7 \text{ AND } 13}$

The following theorem is an extension of Andrews' congruences (1.4), (1.5), and (1.6).

Theorem 2.1. *Let $t = 5, 7$ or 13 . Then*

$$(2.1) \quad \text{spt}(n) \equiv -2\left(n + \frac{t^2-1}{24}\right)p(n) \pmod{t},$$

provided $1 - 24n$ is not a quadratic residue mod t .

Proof. We proceed by considering the three cases $t = 5, 7, 13$ separately. In each case we find the result by reducing a known exact relation between rank and crank moments $[8] \pmod{t}$.

$t = 5$. We need [8, (5.7),p.359]:

$$\begin{aligned} N_6(n) = & \frac{2}{33}(324n^2 + 69n - 10)M_2(n) + \frac{20}{33}(-45n + 4)M_4(n) + \frac{18}{11}M_6(n) \\ & + (108n^2 - 24n + 1)N_2(n). \end{aligned}$$

Reducing mod 5 we obtain

$$(2.2) \quad N_6(n) \equiv (n^2 + n)M_2(n) + 3M_6(n) + (3n^2 + n + 1)N_2(n) \pmod{5}.$$

Since $m^6 \equiv m^2 \pmod{5}$ we have

$$N_6(n) \equiv N_2(n) \pmod{5}, \quad M_6(n) \equiv M_2(n) \pmod{5}.$$

Thus we may rewrite (2.2) as

$$\begin{aligned} 2n(n+2)N_2(n) & \equiv (n+2)(n+4)M_2(n) \pmod{5} \\ & \equiv 2n(n+2)(n+4)p(n) \pmod{5} \quad (\text{by (1.2)}), \end{aligned}$$

and we have

$$N_2(n) \equiv (n+4)p(n) \pmod{5} \quad \text{if } n \not\equiv 0, 3 \pmod{5}.$$

Thus

$$\begin{aligned} \text{spt}(n) & = np(n) - \frac{1}{2}N_2(n) \\ & \equiv 3(n+1)p(n) \pmod{5} \quad \text{if } n \not\equiv 0, 3 \pmod{5}, \end{aligned}$$

which gives Theorem 2.1 for the case $t = 5$.

$t = 7$. This time we need [8, (5.8),p.360] which is relation between the moments M_2, M_4, M_6, M_8, N_2 and N_8 . Upon reducing this relation mod 7 and using the fact that

$$N_8(n) \equiv N_2(n) \pmod{7}, \quad M_8(n) \equiv M_2(n) \pmod{7},$$

we find that

$$\begin{aligned} 3n(n+1)(n+5)N_2(n) & \equiv 2(n+1)(n+5)(n+6)M_2(n) \pmod{7} \\ & \equiv 4n(n+1)(n+5)(n+6)p(n) \pmod{7} \quad (\text{by (1.2)}), \end{aligned}$$

and

$$N_2(n) \equiv (6n+1)p(n) \pmod{7} \quad \text{if } n \not\equiv 0, 2, 6 \pmod{7}.$$

Thus

$$\begin{aligned} \text{spt}(n) & = np(n) - \frac{1}{2}N_2(n) \\ & \equiv (5n+3)p(n) \pmod{7} \quad \text{if } n \not\equiv 0, 2, 6 \pmod{7}, \end{aligned}$$

which gives Theorem 2.1 for the case $t = 7$.

$t = 13$. For this case we need [8, (5.10), p.360] which is an exact linear relation between the moments

$$N_{14}, N_{12}, N_2, M_2, M_4, M_6, M_8, M_{10}, M_{12}, M_{14}.$$

Reducing this relation mod 13 we find that

$$\begin{aligned} N_{14}(n) &\equiv (4 + 4n + 12n^2 + 4n^3 + 12n^4 + 8n^6)M_2(n) \\ &\quad + (1 + 6n + 4n^2 + 2n^3 + 3n^4 + 5n^5 + 6n^6)N_2(n) + M_{14}(n) \pmod{13}. \end{aligned}$$

Using the fact that

$$N_{14}(n) \equiv N_2(n) \pmod{13}, \quad M_{14}(n) \equiv M_2(n) \pmod{13},$$

we find that

$$\begin{aligned} &7n(n+1)(n+2)(n+5)(n+9)(n+12)N_2(n) \\ &\equiv 8(n+1)(n+2)(n+5)(n+9)^2(n+12)M_2(n) \pmod{13} \\ &\equiv 3n(n+1)(n+2)(n+5)(n+9)^2(n+12)p(n) \pmod{13} \quad (\text{by (1.2)}), \end{aligned}$$

and

$$N_2(n) \equiv (6n+2)p(n) \pmod{13} \quad \text{if } n \not\equiv 0, 1, 4, 8, 11, 12 \pmod{13}.$$

Thus

$$\begin{aligned} \text{spt}(n) &= np(n) - \frac{1}{2}N_2(n) \\ &\equiv (11n+12)p(n) \pmod{13} \quad \text{if } n \not\equiv 0, 1, 4, 8, 11, 12 \pmod{13}, \end{aligned}$$

which gives Theorem 2.1 for the case $t = 13$. □

3. MODULAR FORMS AND HECKE OPERATORS

Let N be a positive integer and k be a nonnegative integer. We let $M_k(N)$ (resp. $S_k(N)$) be the space of entire modular (resp. cusp) forms of weight k with respect to the modular group $\Gamma_0(N)$. If χ is a Dirichlet character mod N we let $M_k(N, \chi)$ (resp. $S_k(N, \chi)$) be the space of entire modular (resp. cusp) forms of weight k and character χ with respect to the modular group $\Gamma_0(N)$. We define half-integral weight modular forms in the sense of Shimura [29]. If N is divisible by 4 we let $M_{k+\frac{1}{2}}(N, \chi)$ (resp. $S_{k+\frac{1}{2}}(N, \chi)$) be the space of modular (resp. cusp) forms of weight $k + \frac{1}{2}$ and character χ with respect to the modular group $\Gamma_0(N)$.

Let $GL_2^+(\mathbb{R})$ denote the group of all real 2×2 matrices with positive determinant. $GL_2^+(\mathbb{R})$ acts on the complex upper half plane \mathcal{H} by linear fractional transformations. As in [29] we let G denote the set of ordered pairs $\alpha, \phi(\tau)$, where $\alpha \in GL_2^+(\mathbb{R})$ with last row $(c \ d)$, and ϕ is a holomorphic function on \mathcal{H} such that

$$\phi^2(\tau) = s \det \alpha^{-1/2} (c\tau + d),$$

where $|s| = 1$. G is a group with multiplication

$$(\alpha, \phi(\tau))(\beta, \psi(\tau)) = (\alpha\beta, \phi(\beta\tau)\psi(\tau)).$$

For a holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ and $\xi = (\alpha, \phi(\tau)) \in G$ we define

$$f \mid_{k+\frac{1}{2}} \xi = f \mid \xi = \phi(\tau)^{-2k-1} f(\alpha\tau).$$

For a prime ℓ the Hecke operator $T_{k,N}(\ell^2) = T(\ell^2)$ (as defined in [29]) maps $M_{k+\frac{1}{2}}(N, \chi)$ to itself. If $f = \sum_{n=0}^{\infty} a(n)q^n$ then $f \mid T(\ell^2) = \sum_{n=0}^{\infty} c(n)q^n$ where

$$(3.1) \quad c(n) = a(\ell^2 n) + \chi(\ell) \left(\frac{(-1)^k n}{\ell} \right) \ell^{k-1} a(n) + \chi(\ell^2) \ell^{2k-1} a(n/\ell^2).$$

We note the convention that $a(x) = 0$ if x is not a nonnegative integer. Let

$$W_N = \left(\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{1/4} \sqrt{-i\tau} \right) \in G.$$

Then the *Fricke involution* is given by $f \mapsto f \mid_{k+\frac{1}{2}} W_N$

The Dedekind eta-function is defined by

$$\eta(\tau) = \exp(\pi i \tau / 12) \prod_{n=1}^{\infty} (1 - \exp(2\pi i n \tau)) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Then for $\tau \in \mathcal{H}$

$$(3.2) \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$$

and $\eta(24\tau)$ is $\frac{1}{2}$ weight cusp form in $S_{\frac{1}{2}}(576, \chi_{12})$ where

$$\chi_{12}(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. *Let $1 \leq r \leq 23$ with $(r, 6) = 1$, m be an even integer nonnegative integer, and ℓ be a prime, $\ell > 3$. Define*

$$\mathcal{C}_{r,k} := \{\eta^r(24\tau)F(24\tau) : F \in M_m(1)\} \subset S_{\frac{r}{2}+m}(576, \chi_{12}).$$

Then the Hecke operator $T(\ell^2)$ maps $\mathcal{C}_{r,k}$ to $\mathcal{C}_{r,k}$.

Proof. Suppose

$$g(\tau) = \eta^r(24\tau)F(24\tau),$$

where $F \in M_m(1)$ and ℓ is a prime, $\ell > 3$. Then $g(\tau) \in S_{k+\frac{1}{2}}(576, \chi_{12})$ where $k = m + \frac{r-1}{2}$. By [17, (6)] we have

$$g \mid W_{576} \mid T(\ell^2) = g \mid T(\ell^2) \mid W_{576},$$

since $(\ell, 576) = 1$ and χ_{12} is a real character. Since $F \in M_m(1)$ we have

$$(3.3) \quad F(-1/\tau) = \tau^m F(\tau).$$

Hence using (3.2) and (3.3) we find that

$$\begin{aligned} g \mid W_{576} &= \left(\sqrt{24} \sqrt{-i\tau} \right)^{-2k-1} g(-1/(576\tau)) \\ &= \left(\sqrt{24} \sqrt{-i\tau} \right)^{-2k-1} \eta^r(-1/(24\tau)) F(-1/(24\tau)) \\ &= i^m g. \end{aligned}$$

Now let

$$H(\tau) = \frac{g \mid T(\ell^2)}{\eta^r(24\tau)}.$$

Then

$$\begin{aligned} (3.4) \quad H(-1/(576\tau)) &= \left(\sqrt{24}\sqrt{-i\tau}\right)^{2k+1} \frac{g \mid T(\ell^2) \mid W_{576}}{\eta^2(-1/(24\tau))} \\ &= \left(\sqrt{24}\sqrt{-i\tau}\right)^{2k+1} \frac{g \mid W_{576} \mid T(\ell^2)}{(-24i\tau)^{r/2}\eta^2(-1/(24\tau))} \\ &= i^m (-24i\tau)^{k+\frac{1-r}{2}} \frac{g \mid T(\ell^2)}{\eta^r(24\tau)} \\ &= (24\tau)^m H(\tau). \end{aligned}$$

We note the exponents of q in the q -expansion of $g(\tau)$ are congruent to $r \pmod{24}$. Since $\ell^2 \equiv 1 \pmod{24}$ we see by (3.1) that the exponents of q in the q -expansion of $g(\tau) \mid T(\ell^2)$ are also congruent to $r \pmod{24}$. Since

$$\eta^r(24\tau) = q^r + \cdots$$

and $\eta(\tau)$ is nonzero in \mathcal{H} we see that

$$H(\tau) = K(24\tau),$$

for some function $K(\tau)$ holomorphic function on \mathcal{H} that satisfies

$$K(\tau + 1) = K(\tau).$$

From (3.4) we have

$$K(-1/\tau) = H(-1/(24\tau)) = \tau^m H(\tau/24) = \tau^m K(\tau).$$

Hence $K(\tau) \in M_m(1)$ and

$$g \mid T(\ell^2) = \eta^r(24\tau)K(24\tau) \in \mathcal{C}_{r,k}.$$

□

Corollary 3.2. *Let $1 \leq r \leq 23$ with $(r, 6) = 1$, and m be an even integer nonnegative integer. If $\dim M_m(1) = 1$ and $0 \neq F(\tau) \in M_m(1)$, then the function*

$$g(\tau) = \eta^r(24\tau)F(24\tau)$$

is a Hecke eigenform for $S_{\frac{r}{2}+m}(576, \chi_{12})$.

We define the slash operator for modular forms of integer weight. Let $k \in \mathbb{Z}$. For a holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ and $\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ we define

$$f \mid_k \alpha = f \mid \alpha = (\det \alpha)^{\frac{k}{2}} (c\tau + d)^{-k} f(\alpha\tau).$$

For m a positive integer we define the operator $U(m)$ by its action on formal power series

$$\left(\sum_{n=0}^{\infty} a(n)q^n \right) \mid U(m) = \sum_{n=0}^{\infty} a(mn)q^n.$$

We will also need the Hecke operator $T(\ell)$ on $M_k(1)$. For a positive integer m the Hecke operator $T(m)$ (as defined in [5]) maps $M_k(1)$ and $S_k(1)$ to themselves. Let ℓ be prime. If $f = \sum_{n=0}^{\infty} a(n)q^n$ then $f \mid T(\ell) = \sum_{n=0}^{\infty} c(n)q^n$ where

$$c(n) = a(\ell n) + \ell^{k-1}a(n/\ell).$$

Hence, if the coefficients $a(n)$ are integers we have

$$f \mid T(\ell) \equiv f \mid U(\ell) \pmod{\ell^{k-1}}.$$

Following Chua [18] we define

$$(3.5) \quad h_\ell(\tau) = (\eta(\tau)\eta(\ell\tau))^{\ell-1},$$

for $\ell > 3$ prime. The following proposition is basically an extension of a result of Chua [18]. The $F = 1$ case follows from Lemmas 2.1 and 2.2 in [18].

Proposition 3.3. *Let $\ell > 3$ be prime, k be a nonnegative even integer. If $F(\tau) \in M_k(1)$ then*

$$(3.6) \quad h_\ell(\tau)F(\tau) \mid U(\ell) + (-1)^{(\ell-1)/2}\ell^{k+(\ell-1)/2-1}h_\ell(\tau)F(\ell\tau) \in S_{k+\ell-1}(1).$$

Proof. Suppose $\ell > 3$ is prime, k is a nonnegative even integer, and $F(\tau) \in M_k(1)$. Chua [18] showed that $h_\ell(\tau) \in M_{\ell-1}(\ell)$. Actually $h_\ell(\tau) \in S_{\ell-1}(\ell)$ since clearly h_ℓ is zero at the cusp $i\infty$ and zero at the cusp 0 by (3.8) below. Hence

$$g_\ell(\tau) = h_\ell(\tau)F(\tau) \in S_{k+\ell-1}(\ell).$$

By [10, Lemma 17(iii), p.144],

$$(3.7) \quad g_\ell(\tau) \mid U(\ell) + \ell^{k+\ell-1}g_\ell(\tau) \mid W_\ell \in S_{k+\ell-1},$$

where

$$W_\ell = \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}.$$

By (3.2) we have

$$(3.8) \quad h_\ell\left(-\frac{1}{\ell\tau}\right) = (-1)^{(\ell-1)/2}\ell^{(\ell-1)/2}\tau^{\ell-1}h_\ell(\tau).$$

Since $F(\tau) \in M_k(1)$, we have

$$(3.9) \quad F\left(-\frac{1}{\ell\tau}\right) = \ell^k\tau^kF(\ell\tau).$$

Now by (3.8) and (3.9) we have

$$(3.10) \quad g_\ell \mid_{k+\ell-1} W_\ell = \ell^{-(k+\ell-1)/2}\tau^{-k-\ell+1}g_\ell\left(-\frac{1}{\ell\tau}\right) = (-1)^{(\ell-1)/2}\ell^{k/2}h_\ell(\tau)F(\ell\tau).$$

The result (3.6) follows from (3.7) and (3.10). \square

Let $F \in M_k(1)$ and suppose the q -expansion of $F(\tau)$ has integer coefficients. We call such a form an *integral* modular form. We define $p(F, n)$ by

$$\sum_{n=0}^{\infty} p(F, n)q^n = \frac{F(\tau)}{\prod_{n=1}^{\infty} (1 - q^n)}.$$

We prove a straightforward generalization of a theorem due to Chua [18, Theorem 1.1]. We note that Chua's result was extended to prime power moduli by Ahlgren and Boylan [1].

Theorem 3.4. Suppose $\ell > 3$ is prime, k is a nonnegative even integer, and $F(\tau) \in M_k(1)$ is an integral modular form. Define $1 \leq \beta_\ell \leq \ell - 1$ such that $24\beta_\ell \equiv 1 \pmod{\ell}$ and let

$$r_\ell = \frac{24\beta_\ell - 1}{\ell}, \quad \lambda_\ell = \frac{\ell^2 + 24\beta_\ell - 1}{24\ell}.$$

Then

$$\sum_{n=0}^{\infty} p(F, \ell n + \beta_\ell) q^{24n+r_\ell} \equiv \eta^{r_\ell}(24\tau) G_{\ell,F}(24\tau) \pmod{\ell},$$

for some integral modular form $G_{\ell,F}(\tau) \in M_{k+\ell-1-12\lambda_\ell}(1)$.

Proof. Suppose $\ell > 3$ is prime, k is a nonnegative even integer, and $F(\tau) \in M_k(1)$.

$$\begin{aligned} \sum_{n=0}^{\infty} p(F, n) q^{24n-1} &= \frac{F(24\tau)}{\eta(24\tau)} \\ &\equiv \frac{F(24\tau)}{\eta(24\tau)} \frac{\eta^\ell(24\tau)}{\eta(24\ell\tau)} \pmod{\ell} \\ &\equiv \frac{h_\ell(24\tau) F(24\tau)}{\eta^\ell(24\ell\tau)} \pmod{\ell}, \end{aligned}$$

where $h_\ell(\tau)$ is defined in (3.5). Applying the $U(\ell)$ operator to both sides we obtain

$$\sum_{n=0}^{\infty} p(F, \ell n + \beta_\ell) q^{24n+r_\ell} \equiv \frac{h_\ell(24\tau) F(24\tau) | U(\ell)}{\eta^\ell(24\tau)} \pmod{\ell}.$$

By Proposition 3.3 there is cusp form $j_\ell(\tau) \in M_{k+\ell-1}$ such that

$$\begin{aligned} (3.11) \quad j_\ell(\tau) &= h_\ell(\tau) F(\tau) | U(\ell) + (-1)^{(\ell-1)/2} \ell^{k+(\ell-1)/2-1} h_\ell(\tau) F(\ell\tau) \\ &= \left(c_1 q^{(\ell^2-1)/24} + \dots \right) | U(\ell) + O(q^{(\ell^2-1)/24}), \\ &= c_2 q^{\lambda_\ell} + \dots \end{aligned}$$

for some constants c_1, c_2 since for $\ell \geq 5$, $\lambda_\ell \leq \frac{\ell^2-1}{24}$. It follows that

$$(3.12) \quad j_\ell(\tau) = \Delta^{\lambda_\ell}(\tau) G_{\ell,F}(\tau),$$

for some $G_{\ell,F}(\tau) \in M_{k+\ell-1-12\lambda_\ell}(1)$. Here as usual

$$\Delta(\tau) = \eta^{24}(\tau).$$

Since $\ell \geq 5$, $k + (\ell - 1)/2 - 1 \geq 1$, and

$$h_\ell(\tau) F(\tau) | U(\ell) \equiv \Delta^{\lambda_\ell}(\tau) G_{\ell,F}(\tau) \pmod{\ell},$$

by (3.11) and (3.12). Since $(\ell, 24) = 1$ we have

$$\sum_{n=0}^{\infty} p(F, \ell n + \beta_\ell) q^{24n+r_\ell} \equiv \frac{\Delta^{\lambda_\ell}(24\tau) G_{\ell,F}(24\tau)}{\eta^\ell(24\tau)} \equiv \eta^{r_\ell}(24\tau) G_{\ell,F}(24\tau) \pmod{\ell},$$

since $24\lambda_\ell - \ell = r_\ell$. □

4. CONGRUENCES FOR $\text{spt}(n) \pmod{11}$

We need some results for the crank mod 11. For $t \geq 2$ and $0 \leq r \leq t-1$ let $M(r, t, n)$ denote the number of partitions of n with crank congruent to $r \pmod{t}$. Then for $t = 5$, $t = 7$, or $t = 11$

$$M(r, t, n) = \frac{1}{t} p(n), \quad 0 \leq r \leq t-1;$$

for all n satisfying $24n \equiv 1 \pmod{t}$. See [4], [22]. These combinatorial results immediately imply Ramanujan's partition congruences (1.7), (1.8), (1.9). Here we need the $t = 11$ case

$$(4.1) \quad M(r, 11, 11n+6) = \frac{1}{11} p(11n+6), \quad 0 \leq r \leq 10.$$

Let

$$E(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

For $r \geq 1$ define $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n) q^n = E(q)^r = \prod_{n=1}^{\infty} (1 - q^n)^r,$$

so that

$$\sum_{n=1}^{\infty} p_{23}(n-1) q^n = q E(q)^{23} = q \prod_{n=1}^{\infty} (1 - q^n)^{23}.$$

We need [8, (5.24), pp.362-3] which is an exact linear relation between rank-crank moments and the function $p_{23}(n-1)$. Upon reducing this relation mod 11 and using the fact that

$$N_{12}(n) \equiv N_2(n) \pmod{11}, \quad M_{12}(n) \equiv M_2(n) \pmod{11},$$

we find that

$$\begin{aligned} p_{23}(11n+5) &\equiv 4N_2(11n+6) + 2M_2(11n+6) \\ &\quad + M_4(11n+6) + M_6(11n+6) + 10M_8(11n+6) \pmod{11}. \end{aligned}$$

By (4.1) we have

$$\begin{aligned} M_k(11n+6) &\equiv \sum_{m=1}^{10} m^k M(m, 11, 11n+6) \pmod{11} \\ &\equiv M(1, 11, 11n+6) \sum_{m=1}^{10} m^k \pmod{11} \\ &\equiv 0 \pmod{11}, \quad \text{if } k = 2, 4, 6, 8. \end{aligned}$$

Hence

$$(4.2) \quad N_2(11n+6) \equiv 3p_{23}(11n+5) \pmod{11}$$

and

$$\begin{aligned} \text{spt}(11n+6) &= (11n+6)p(11n+6) - \frac{1}{2}N_2(11n+6) \\ &\equiv 4p_{23}(11n+5) \pmod{11}. \end{aligned}$$

Now by the Children's Binomial Theorem we have

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - q^n)^{23} &= \prod_{n=1}^{\infty} (1 - q^n)^{22} \prod_{n=1}^{\infty} (1 - q^n) \\ &\equiv \prod_{n=1}^{\infty} (1 - q^{11n})^2 \prod_{n=1}^{\infty} (1 - q^n) \pmod{11}. \end{aligned}$$

We need Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Since $n(3n-1)/2 \equiv 5 \pmod{11}$ if and only if $n \equiv 2 \pmod{11}$ we find that

$$\begin{aligned} (4.3) \quad \sum_{n=0}^{\infty} p_{23}(11n+5)q^n &\equiv \prod_{n=1}^{\infty} (1 - q^n)^2 \prod_{n=1}^{\infty} (1 - q^{11n}) \pmod{11} \\ &\equiv \prod_{n=1}^{\infty} (1 - q^n)^{13} \pmod{11}. \end{aligned}$$

Hence

$$\text{spt}(11n+6) \equiv 4p_{13}(n) \pmod{11}.$$

Now let

$$F_{13}(\tau) = \eta^{13}(24\tau) = \sum_{n \geq 13} b_{13}(n)q^n \in S_{\frac{13}{2}}(576, \chi_{12}),$$

where

$$b_{13}(n) = p_{13}\left(\frac{n-13}{24}\right).$$

Then by Theorem 3.1, F_{13} is a Hecke eigenform so

$$(4.4) \quad F_{13} \mid T(\ell^2) = \lambda_{\ell} F_{13}, \quad (\ell > 3),$$

where $\lambda_{\ell} \in \mathbb{Z}$. We look for the smallest eigenvalue which is a multiple of 11. After some calculation we find that

$$\lambda_{19} = -2901404 = -2^2 \cdot 11 \cdot 23 \cdot 47 \cdot 61 \equiv 0 \pmod{11},$$

By (4.4) and (3.1) we have

$$b_{13}(19^3 m) = \lambda_{19} b_{13}(19m) \equiv 0 \pmod{11} \quad \text{when } (m, 19) = 1.$$

We want

$$19^3 m \equiv 13 \pmod{24}, \quad \text{and } (m, 19) = 1,$$

and take

$$m = 24 \cdot 19k + 7, \quad n = 19^3(24 \cdot 19k + 7)$$

so that

$$\frac{n-13}{24} = 19^4 k + 2000.$$

Hence

$$p_{13}(19^4 n + 2000) \equiv 0 \pmod{11},$$

and

$$\text{spt}(19^4 \cdot 11n + 22006) \equiv 0 \pmod{11}.$$

5. EXPLICIT CONGRUENCES FOR THE RANK MOD 11

In this section we find explicit congruences for the rank mod 11. From (4.2) and (4.3) we find that the generating function for $N_2(11n+6)$ is congruent to $3E(q)^{13} \pmod{11}$. In the following theorem we give congruences mod 11 for other rank moments $N_{2k}(11n+6)$ in terms of half-integer weight cusp forms.

Following [8] and Ramanujan [27, p.163] we define

$$\Phi_j = \Phi_j(q) = \sum_{n=1}^{\infty} \frac{n^j q^n}{1-q^n} = \sum_{m,n \geq 1} n^j q^{nm} = \sum_{n=1}^{\infty} \sigma_j(n) q^n,$$

for $j \geq 1$ odd and where $\sigma_j(n) = \sum_{d|n} d^j$. For $n \geq 2$ even we define the Eisenstein series

$$E_n(\tau) = 1 - \frac{2n}{B_n} \Phi_{n-1}(q),$$

where $q = \exp(2\pi i\tau)$, $\Im\tau > 0$ and B_n is the n -th Bernoulli number. We note that $E_n(\tau) \in M_n(1)$ for $n \geq 4$.

Theorem 5.1.

$$(5.1) \quad \sum_{n=0}^{\infty} N_2(11n+6) q^n \equiv 3E^{13}(q) \pmod{11},$$

$$(5.2) \quad \sum_{n=0}^{\infty} N_4(11n+6) q^n \equiv 7E^{13}(q) \pmod{11},$$

$$(5.3) \quad \sum_{n=0}^{\infty} N_6(11n+6) q^n \equiv E^{13}(q) (4 + E_4(\tau)) \pmod{11},$$

$$(5.4) \quad \sum_{n=0}^{\infty} N_8(11n+6) q^n \equiv E^{13}(q) (5 + 6E_4(\tau) + 6E_6(\tau)) \pmod{11},$$

$$(5.5) \quad \sum_{n=0}^{\infty} N_{10}(11n+6) q^n \equiv E^{13}(q) (5 + 4E_4(\tau) + 6E_6(\tau) + 6E_4^2(\tau)) \pmod{11}.$$

It is clear the each rank moment $N_{2k}(11n+6) \pmod{11}$ can be written in terms the rank functions $N(r, 11, 11n+6)$, $(0 \leq r \leq 5)$. These relations may be inverted to find each $N(r, 11, 11n+6)$ in terms of rank moments mod 11. We note that only $r \leq 5$ is needed since $N(-m, n) = N(m, n)$ and $N(11-r, 11, n) = N(r, 11, n)$.

Corollary 5.2.

$$\begin{aligned} \sum_{n=0}^{\infty} N(0, 11, 11n+6) q^n &\equiv E^{13}(q) (6 + 7E_4(\tau) + 5E_6(\tau) + 5E_4^2(\tau)) \pmod{11}, \\ \sum_{n=0}^{\infty} N(1, 11, 11n+6) q^n &\equiv E^{13}(q) (9 + 10E_6(\tau) + 5E_4^2(\tau)) \pmod{11}, \\ \sum_{n=0}^{\infty} N(2, 11, 11n+6) q^n &\equiv E^{13}(q) (4 + 3E_6(\tau) + 5E_4^2(\tau)) \pmod{11}, \end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} N(3, 11, 11n+6) q^n &\equiv E^{13}(q) (8 + 4E_4(\tau) + 6E_6(\tau) + 5E_4^2(\tau)) \pmod{11}, \\
\sum_{n=0}^{\infty} N(4, 11, 11n+6) q^n &\equiv E^{13}(q) (2 + 7E_4(\tau) + 8E_6(\tau) + 5E_4^2(\tau)) \pmod{11}, \\
\sum_{n=0}^{\infty} N(5, 11, 11n+6) q^n &\equiv E^{13}(q) (7 + 2E_4(\tau) + 9E_6(\tau) + 5E_4^2(\tau)) \pmod{11}.
\end{aligned}$$

Although the half-integer modular forms appearing in the previous theorem have different weights each one is a Hecke eigenform (in its corresponding space) in view of Corollary 3.2. As a result we find following rank congruences.

Corollary 5.3.

$$(5.6) \quad N(r, 11, 5^4 \cdot 11 \cdot 19^4 \cdot n + 4322599) \equiv 0 \pmod{11},$$

$$(5.7) \quad N(r, 11, 11^2 \cdot 19^4 \cdot n + 172904) \equiv 0 \pmod{11},$$

for all $0 \leq r \leq 10$.

Before we can prove Theorem 5.1 we need to recall some results on rank and crank moments from [8]. We define the moment generating functions.

$$\begin{aligned}
R_k(q) &= \sum_{n \geq 0} N_k(n) q^n, \\
C_k(q) &= \sum_{n \geq 0} M_k(n) q^n,
\end{aligned}$$

for k even. Define

$$P = P(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)},$$

for $|q| < 1$. In [8] we proved the following recurrence for crank moments

$$C_{2n} = 2 \sum_{j=1}^{n-1} \binom{2n-1}{2j-1} \Phi_{2j-1} C_{2n-2j} + 2\Phi_{2n-1} P,$$

which implies that there are integers $\alpha_{a_1, a_2, \dots, a_n}$ such that

$$(5.8) \quad C_{2n} = 2P \sum_{a_1 + 2a_2 + \dots + na_n = n} \alpha_{a_1, a_2, \dots, a_n} \Phi_1^{a_1} \Phi_3^{a_2} \dots \Phi_{2n-1}^{a_n},$$

In [8] we obtained the following identity

$$\begin{aligned}
(5.9) \quad &\sum_{i=0}^{k-1} \binom{2k}{2i} \sum_{\substack{\alpha + \beta + \gamma = 2k - 2i \\ \alpha, \beta, \gamma \text{ even} \geq 0}} \binom{2k-2i}{\alpha, \beta, \gamma} C_{\alpha} C_{\beta} C_{\gamma} P^{-2} - 3(2^{2k-1} - 1) C_2 \\
&= \frac{1}{2}(2k-1)(2k-2)R_{2k} + 6 \sum_{i=1}^{k-1} \binom{2k}{2i} (2^{2i-1} - 1) \delta_q(R_{2k-2i})
\end{aligned}$$

$$+ \sum_{i=1}^{k-1} \left[\binom{2k}{2i+2} (2^{2i+1} - 1) - 2^{2i} \binom{2k}{2i+1} + \binom{2k}{2i} \right] R_{2k-2i}.$$

Here δ_q is the differential operator

$$\delta_q = q \frac{d}{dq}.$$

Proof of Theorem 5.1. As noted above, the first congruence (5.1) follows from (4.2) and (4.3). To attack (5.2)–(5.5), we first use (5.9), and (5.8) rewriting each Φ_{2j-1} in terms of $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$.

We define the operator U_{11}^* which acts on q -series by

$$U_{11}^* \left(\sum_{n=0}^{\infty} a(n) q^n \right) = \sum_{n=0}^{\infty} a(11n+6) q^n.$$

We find that

$$(5.10) \quad U_{11}^* (R_4) \equiv U_{11}^* (6R_2 + PH^{(4)}) \pmod{11},$$

$$(5.11) \quad U_{11}^* (R_6) \equiv U_{11}^* (5R_2 + PH^{(6)}) \pmod{11},$$

$$(5.12) \quad U_{11}^* (R_8) \equiv U_{11}^* (9R_2 + PH^{(8)}) \pmod{11},$$

$$(5.13) \quad U_{11}^* (R_{10}) \equiv U_{11}^* (9R_2 + \Psi + PH^{(10)}) \pmod{11},$$

where

$$H^{(4)} = 9 + 5E_2 + 10E_4 + 9E_2^2$$

$$H^{(6)} = 2 + 7E_2 + 7E_4 + 3E_2^2 + 8E_6 + 8E_4E_2 + 9E_2^3$$

$$H^{(8)} = 6E_2 + E_2^2 + 6E_4 + 10E_2^3 + 4E_6 + 4E_4E_2 + 9E_2^4 + 5E_2^2E_4 + 10E_6E_2$$

$$H^{(10)} = 8 + 4E_2 + 8E_2^2 + 4E_4 + 3E_2^3 + 10E_6 + 10E_4E_2 + 9E_2^4 + 5E_2^2E_4 + 10E_6E_2 + 4E_2^5 + 10E_4E_6 + 5E_2^2E_6 + 9E_2^3E_4$$

and

$$(5.14) \quad \Psi = \frac{1}{11} P(1 - E_4E_6).$$

We call a function 11-integral if the coefficients in its q -expansion are rational numbers with bounded denominators relatively prime to 11. We note that all functions involved are 11-integral so that each congruence is well-defined. For example, Ψ is 11-integral since

$$(5.15) \quad E_4E_6 = E_{10} = 1 - 264\Phi_9 \equiv 1 \pmod{11}.$$

For each m we write

$$H^{(m)} = H_0^{(m)} + H_2^{(m)} + H_4^{(m)} + H_6^{(m)} + H_8^{(m)},$$

where each $H_j^{(m)}$ is congruent mod 11 to an 11-integral modular form in $M_{\kappa(m,j)}(1)$. Here the weight $\kappa = \kappa(m, j) \equiv j \pmod{10}$. Here we have used [28, Theorem 2, p.22]. We note that

$$E_2 \equiv E_{12} \pmod{11}, \quad \text{and} \quad E_{10} \equiv 1 \pmod{11}.$$

By Theorem 3.4, each $U_{11}^*(PH_j^{(m)})$ is congruent to a function $E^{13}(q)G_{m,j}(\tau)$ where $G_{m,j}(\tau)$ to a modular form of weight $\kappa(m, j) - 2$. If $H_j^{(m)}$ is nonzero we prove a congruence for $U_{11}^*(PH_j^{(m)})$ by checking enough coefficients of q^n (i.e. $n \leq \lfloor \frac{\kappa(m,j)-2}{12} \rfloor \leq 5$). This is a standard argument. See for example [28], [30]. For example, for $m = 10$ we have

$$H^{(10)} = H_0^{(10)} + H_2^{(10)} + H_4^{(10)} + H_6^{(10)} + H_8^{(10)}$$

where

$$H_0^{(10)} = 8 + 4E_2^5 + 10E_4E_6 + 5E_2^2E_6 + 9E_2^3E_4 \quad (\kappa = 60)$$

$$H_2^{(10)} = 4E_2 \quad (\kappa = 12)$$

$$H_4^{(10)} = 8E_2^2 + 4E_4 \quad (\kappa = 24)$$

$$H_6^{(10)} = 3E_2^3 + 10E_6 + 10E_4E_2 \quad (\kappa = 36)$$

$$H_8^{(10)} = 9E_2^4 + 5E_2^2E_4 + 10E_6E_2 \quad (\kappa = 48)$$

We prove the following congruences

$$U_{11}^*(PH_0^{(10)}) \equiv 4E^{13}(q)E_4^2 \pmod{11},$$

$$U_{11}^*(PH_2^{(10)}) \equiv 0 \pmod{11},$$

$$U_{11}^*(PH_4^{(10)}) \equiv 0 \pmod{11},$$

$$U_{11}^*(PH_6^{(10)}) \equiv 4E^{13}(q)E_4 \pmod{11},$$

$$U_{11}^*(PH_8^{(10)}) \equiv 6E^{13}(q)E_6 \pmod{11},$$

by checking enough terms. Hence

$$(5.16) \quad U_{11}^*(PH^{(10)}) \equiv E^{13}(q)(4E_4^2 + 4E_4 + 6E_6) \pmod{11}.$$

Similarly, we find that

$$(5.17) \quad U_{11}^*(PH^{(4)}) \equiv 0 \pmod{11},$$

$$(5.18) \quad U_{11}^*(PH^{(6)}) \equiv E^{13}(q)E_4 \pmod{11},$$

$$(5.19) \quad U_{11}^*(PH^{(8)}) \equiv E^{13}(q)(6E_4 + 6E_6) \pmod{11}.$$

To prove (5.5) we also need to compute $U_{11}^*(\Psi) \pmod{11}$. Since

$$\left(\frac{E(q)^{11}}{E(q^{11})} \right)^{11} \equiv 1 \pmod{11^2},$$

we have

$$(5.20) \quad qU_{11}^*(P(1 - E_4E_6)) \equiv \frac{\Delta^5(1 - E_4E_6) | U_{11}}{E(q)^{11}} \pmod{11^2}.$$

Now $\Delta^5 \in S_{60}(1)$ and $E_4 E_6 \Delta^5 \in S_{70}(1)$. The set

$$\mathcal{B}_1 = \{\Delta E_4^{12}, \Delta^2 E_6^6, \Delta^3 E_6^4, \Delta^4 E_6^2, \Delta^5\}$$

is a basis for $S_{60}(1)$ and multiplying each element by $E_4 E_6$ gives a basis \mathcal{B}_2 for $S_{70}(1)$. We have

$$\begin{aligned} \Delta^5 \mid U_{11} &\equiv \Delta^5 \mid_{60} T(11) \pmod{11^2}, \\ E_4 E_6 \Delta^5 \mid U_{11} &\equiv \Delta^5 \mid_{72} T(11) \pmod{11^2}. \end{aligned}$$

We express $\Delta^5 \mid_{60} T(11)$ in terms of \mathcal{B}_1 , $E_4 E_6 \Delta^5 \mid_{72} T(11)$ in terms of \mathcal{B}_2 , and reduce mod 11^2 to find

$$\begin{aligned} \Delta^5(1 - E_4 E_6) \mid U_{11} &\equiv 11\Delta(1 + E_4 E_6)(E_4^{12} + 10\Delta^4 + 6\Delta^2 E_6^4 + 7\Delta^3 E_6^2 + 9\Delta E_6^6) \\ &\equiv 22E_4^2 \Delta \pmod{11^2}, \end{aligned}$$

by using

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

and (5.15) It follows from (5.20) that

$$(5.21) \quad U_{11}^*(\Psi) \equiv 2E^{13}(q)E_4^2 \pmod{11}.$$

Finally, (5.5) follows from (5.1), (5.13), (5.16) and (5.21). This completes the proof of Theorem 5.1.

Proof of Corollary 5.3. From Corollary 5.2 we see that for each $0 \leq r \leq 10$, that there are integers a_r, b_r, c_r , and d_r such

$$(5.22) \quad \sum_{n=0}^{\infty} N(r, 11, \frac{1}{24}(11n+1))q^n \equiv \eta^{13}(24\tau)(a_r + b_r E_4(24\tau) + c_r E_6(24\tau) + d_r E_4^2(24\tau)) \pmod{11}.$$

By Corollary 3.2 we note that each of the functions

$$\eta^{13}(24\tau), \quad \eta^{13}(24\tau)E_4(24\tau), \quad \eta^{13}(24\tau)E_6(24\tau), \quad \eta^{13}(24\tau)E_4^2(24\tau),$$

is a Hecke eigenform in its corresponding space; i.e. $S_w(576, \chi_{12})$ for $w = \frac{13}{2}, w = \frac{13}{2}, w = \frac{21}{2}, w = \frac{25}{2}$, and $w = \frac{29}{2}$ respectively. For each form we find eigenvalues divisible by 11. For the forms $\eta^{13}(24\tau)$, $\eta^{13}(24\tau)E_4(24\tau)$, and $\eta^{13}(24\tau)E_4^2(24\tau)$ we find that λ_{19} is divisible by 11. For $\eta^{13}(24\tau)E_6(24\tau)$ we find that the eigenvalue λ_5 is divisible by 11. From (5.22) and (3.1) it follows that

$$N(r, 11, \frac{1}{24}(5^3 \cdot 11 \cdot 19^3 n + 1)) \equiv 0 \pmod{11}$$

for each r provided that $(n, 5) = (n, 19) = 1$. We replace n by $5 \cdot 19 \cdot 24 \cdot n + c$ where $(c, 5) = (c, 19) = 1$ and $5^3 \cdot 11 \cdot 19^3 c \equiv -1 \pmod{24}$. The smallest such c is $c = 11$. This gives

$$N(r, 11, 5^4 \cdot 11 \cdot 19^4 n + 4322599) \equiv 0 \pmod{11},$$

which is (5.6). We claim that

$$(5.23) \quad \eta^{13}(24\tau)F(24\tau) \mid U_{11} \equiv 0 \pmod{11},$$

for $F(\tau) = E_4(\tau), E_6(\tau)$ or $E_4^2(\tau)$. We observe that

$$\eta^{13}(24\tau)F(24\tau) \equiv \frac{\Delta(24\tau)F(24\tau)}{\eta(24 \cdot 11\tau)} \pmod{11}$$

so that

$$\eta^{13}(24\tau)F(24\tau) \mid U_{11} \equiv \frac{\Delta(24\tau)F(24\tau) \mid U_{11}}{\eta(24\tau)} \pmod{11}.$$

Let $F = E_4$ so that $\Delta(\tau)E_4(\tau) \in S_{16}(1)$. We easily find that

$$\Delta(\tau)E_4(\tau) \mid U_{11} \equiv \Delta(\tau)E_4(\tau) \mid T(11) \equiv 0 \pmod{11},$$

just by checking that the first coefficient is divisible by 11 since $\dim S_{16} = 1$ and $\Delta(\tau)E_4(\tau) \mid T(11) \in S_{16}(1)$. This proves (5.23) for the case $F = E_4$. The other two cases are analogous. Now (5.23) together with the fact that the eigenvalue λ_{19} for $\eta^{13}(24\tau)$ is divisible by 11 implies that

$$N(r, 11, \frac{1}{24}(11^2 \cdot 19^3 n + 1)) \equiv 0 \pmod{11}$$

for each r provided that $(n, 19) = 1$. We replace n by $19 \cdot 24 \cdot n + c$ where $(c, 19) = 1$ and $11^2 \cdot 19^3 c \equiv -1 \pmod{24}$. The smallest such c is $c = 5$. This gives

$$N(r, 11, 11^2 \cdot 19^4 n + 172904) \equiv 0 \pmod{11},$$

which is (5.7).

6. EXPLICIT CONGRUENCES FOR $\text{spt}(n)$

For $\ell > 3$ prime and $\beta \geq 1$ we define

$$\text{SPT}(\ell, \beta) = \sum_{n=0}^{\infty} \text{spt}(\ell n + \beta) q^n.$$

We obtain a number of explicit congruences when $\ell \leq 37$ and $24\beta \equiv 1 \pmod{\ell}$.

Theorem 6.1. *We have*

- (6.1) $\text{SPT}(5, 4) \equiv 0 \pmod{5},$
- (6.2) $\text{SPT}(7, 5) \equiv 0 \pmod{7},$
- (6.3) $\text{SPT}(11, 6) \equiv 4E^{13}(q) \pmod{11},$
- (6.4) $\text{SPT}(13, 6) \equiv 0 \pmod{13},$
- (6.5) $\text{SPT}(17, 5) \equiv 14E^7(q)E_6(\tau) \pmod{17},$
- (6.6) $\text{SPT}(19, 4) \equiv 10E^5(q)E_4^2(\tau) \pmod{19},$
- (6.7) $\text{SPT}(23, 1) \equiv E(q)E_4^3(\tau) + 7qE^{25}(q) \pmod{23},$
- (6.8) $\text{SPT}(29, 23) \equiv 17E^{19}(q)E_6(\tau) \pmod{29},$
- (6.9) $\text{SPT}(31, 22) \equiv 30E^{17}(q)E_4^2(\tau) \pmod{31},$
- (6.10) $\text{SPT}(37, 17) \equiv 12E^{11}(q)E_4^2(\tau)E_6(\tau) \pmod{37},$

Using Theorem 6.1, Proposition 3.1 and Corollary 3.2 we are able to derive a number of explicit congruences for $\text{spt}(n)$.

Theorem 6.2. *We have*

- (6.11) $\text{spt}(11 \cdot 19^4 \cdot n + 22006) \equiv 0 \pmod{11},$
- (6.12) $\text{spt}(7^4 \cdot 17 \cdot n + 243) \equiv 0 \pmod{17},$
- (6.13) $\text{spt}(5^4 \cdot 19 \cdot n + 99) \equiv 0 \pmod{19},$
- (6.14) $\text{spt}(13^4 \cdot 29 \cdot n + 18583) \equiv 0 \pmod{29},$
- (6.15) $\text{spt}(29^4 \cdot 31 \cdot n + 409532) \equiv 0 \pmod{31},$

$$(6.16) \quad \text{spt}(5^4 \cdot 37 \cdot n + 1349) \equiv 0 \pmod{37}.$$

We prove Theorems 6.1 and 6.2 case by case. As usual we let $\ell > 3$ be prime and define $1 \leq \beta_\ell \leq \ell - 1$ such that $24\beta_\ell \equiv 1 \pmod{\ell}$.

We need some results from [8] and [14]. Let n be a positive integer. Define

$$\mathcal{W}_{2n} = \text{Span}\{\Phi_1^a \Phi_3^b \Phi_5^c : 1 \leq a + 2b + 3c \leq n \text{ with } a, b, c \text{ nonnegative integers}\}$$

so that \mathcal{W}_{2n} a vector space of quasimodular forms of bounded weight over \mathbb{Q} . We now define by what we exactly mean by an “ ℓ -integral quasimodular form.” We consider functions $E_2^a(z) F_b(z)$, where $F_b(z) \in M_b(1)$, the coefficients in the q -expansion of $F_b(z)$ are ℓ -integral and have bounded denominators, and a and b are nonnegative integers. We call such a function an ℓ -integral quasimodular form of weight $2a + b$. Let k be a nonnegative integer. In general, an ℓ -integral quasi-modular form of weight k is sum of such functions where $2a + b = k$. We let $\mathcal{X}_{2n} = \mathcal{X}_{2n, \ell}$ denote the subset of ℓ -integral quasimodular forms in \mathcal{W}_{2n} . By Theorems 7.4 and 7.6 in [14]

$$(6.17) \quad R_{2k} - P_k(\delta_q)R_2 \in P\mathcal{X}_{2k, \ell} \subset P\mathcal{W}_{2k},$$

where $P_k(x) \in \mathbb{Z}[x]$, for $2 \leq k \leq \frac{\ell-3}{2}$ and $k = \frac{\ell+1}{2}$. The result for $k = \frac{\ell-1}{2}$ is analogous except a factor ℓ may occur in some denominators. We need a basis for $P\mathcal{W}_{2k}$ in terms of crank moments and cusp forms. Following [8, (5.14)] we let

$$\mathcal{C}_{2k} = \{\delta_q^m(C_{2j}) : 1 \leq j \leq k, j + m \leq k\} \subset P\mathcal{W}_{2k}.$$

We let \mathcal{B}_{2k} be a basis for the space of cusp forms $S_{2k}(1)$, and let

$$\mathcal{S}_{2k} = \bigcup_{\substack{1 \leq j \leq k, \\ 0 \leq m \leq k-j}} \delta_q^m(P\mathcal{B}_{2j}) \subset P\mathcal{W}_{2k}.$$

We

Conjecture 6.3. *For $k \geq 1$ the set*

$$\mathcal{T}_{2k} = \mathcal{C}_{2k} \cup \mathcal{S}_{2k}$$

forms a basis for $P\mathcal{W}_{2k}$ over \mathbb{Q} .

We have confirmed this conjecture for $k \leq 20$ which includes the cases we need. We also note that

$$\dim(P\mathcal{W}_{2k}) = |\mathcal{T}_{2k}|,$$

using [8, (3.32)] and the fact that

$$\dim M_{2k}(1) = 1 + \dim S_{2k}(1).$$

For $\ell > 3$ prime and $\epsilon \in \{-1, 0, 1\}$ we define the operator $U_{\epsilon, \ell}^*$ which acts on q -series by

$$(6.18) \quad U_{\epsilon, \ell}^* \left(\sum a(n)q^n \right) = \sum_{\left(\frac{1-24n}{\ell}\right)=\epsilon} a(n)q^n.$$

In [14, Corollary 7.5] we gave an elementary proof that for $\ell > 3$ prime and $\epsilon = -1$ or 0 there is a $G_\ell \in \mathcal{X}_{\ell+1, \ell}$ such that

$$(6.19) \quad U_{\epsilon, \ell}^*(R_2) \equiv U_{\epsilon, \ell}^*(G_\ell P) \pmod{\ell}.$$

The proof depends on (6.17) and the fact that $R_{\ell+1} \equiv R_2 \pmod{\ell}$, as well some elementary properties of the polynomial $P_{\frac{\ell+1}{2}}(x)$.

Obtaining a congruence mod ℓ for $\text{SPT}(\ell, \beta_\ell)$. To prove Theorem 6.1 for the prime ℓ we follow a series of computational steps:

Step 1. First we compute

$$L_{2k} := R_{2k} - P_k(\delta_q)R_2$$

in terms of the set \mathcal{T}_{2k} for $k = \frac{\ell-1}{2}$ and $k = \frac{\ell+1}{2}$. This step involves the heaviest computation. In each case this calculation was done in MAPLE by computing the coefficients of q^j ($0 \leq j \leq n$ where $n = |\mathcal{T}_{2k}| + 20$) of each function in \mathcal{T}_{2k} as well as the function L_{2k} . These coefficients form a $(n+20) \times n$ matrix A , where each column corresponds to a function in \mathcal{T}_{2k} with the last column corresponding to the function L_{2k} . We used MAPLE to show that $\dim \text{Nul}(A) = 1$ and find a basis vector for this Nullspace. In each case the last component of this basis vector is nonzero thus giving the function L_{2k} as a linear combination of the functions in \mathcal{T}_{2k} since we know that $L_{2k} \in PW_{2k}$ and $|\mathcal{T}_{2k}| = \dim PW_{2k}$.

Step 2. For the identity corresponding to $k = \frac{\ell-1}{2}$ we find coefficients are ℓ -integral except for a factor ℓ in some denominators. We multiply both sides of the identity by ℓ , apply the $U_{0,\ell}^*$ operator (see (6.18)) and reduce mod ℓ . This gives an identity mod ℓ for the second crank moment mod ℓ in terms of certain cusp forms times P . Alternatively, this can be computed by using (1.2).

Step 3. For the identity corresponding to $k = \frac{\ell+1}{2}$ we apply the $U_{0,\ell}^*$ operator and reduce mod ℓ . This gives an identity mod ℓ for the second rank moment mod ℓ in terms of certain cusp forms times P . See (6.19).

Step 4. Using (1.3) and the identities from Steps 2 and 3 we obtain a congruence mod ℓ for $\text{SPT}(\ell, \beta_\ell)$ in terms of $U_{0,\ell}^*(PF)$ where F is a sum of cusp forms of different weights. We use the theory of modular forms mod ℓ and Theorem 3.4 to simplify and obtain the final result. In each case we find that

$$\text{SPT}(\ell, \beta_\ell) \equiv E^{r_\ell}(q)F_\ell(\tau) \pmod{\ell}$$

for some $F_\ell(\tau) \in M_{\frac{1}{2}(\ell-r_\ell)+1}(1) \cap \mathbb{Z}[[q]]$.

Obtaining an explicit congruence mod ℓ for $\text{spt}(n)$. We let

$$K_\ell(\tau) = \eta^{r_\ell}(24\tau)F_\ell(24\tau) \in S_{\frac{\ell+2}{2}}(576, \chi_{12}),$$

$F_\ell(\tau)$ was found in Step 4 (above). In most cases the function $K_\ell(\tau)$ is a Hecke eigenform.

Step 5. Suppose $K_\ell(\tau)$ is a Hecke eigenform; i.e. for each prime $Q > 3$

$$K_\ell(\tau) | T(Q^2) = \lambda_Q K_\ell(\tau),$$

for some integer λ_Q . Find the first prime Q such the eigenvalue $\lambda_Q \equiv 0 \pmod{\ell}$. Now

$$\sum_{n=0}^{\infty} \text{spt}(\ell n + \beta_\ell) q^{24n+r_\ell} = \sum_{n=0}^{\infty} \text{spt}\left(\frac{\ell n + 1}{24}\right) q^n \equiv \eta^{r_\ell}(24\tau)F_\ell(24\tau) \equiv K_\ell(\tau) \pmod{\ell}.$$

It follows from (3.1) that

$$(6.20) \quad \text{spt}\left(\frac{\ell Q^3 n + 1}{24}\right) \equiv 0 \pmod{\ell}$$

provided that $(n, Q) = 1$.

Step 6. For the prime Q found in Step 5 find the smallest integer c , such that $(c, Q) = 1$ and $c \equiv -\ell Q \pmod{24}$. Then in (6.20) we replace n by $24Qn + c$ to obtain

$$(6.21) \quad \text{spt}(\ell Q^4 n + \tfrac{1}{24}(\ell Q^3 c + 1)) \equiv 0 \pmod{\ell}.$$

We note that the constant $\frac{1}{24}(\ell Q^3 c + 1)$ is an integer since $\ell^2 \equiv Q^2 \equiv 1 \pmod{24}$ and $c \equiv -\ell Q \pmod{24}$.

We now prove Theorems 6.1 and 6.2 case by case by following Steps 1–6. All calculations were done using MAPLE. In Section 4 we gave a proof of the $\ell = 11$ case. Here we give the proof of the $\ell = 17$ case in detail and sketch the other cases.

$\ell = 17$. We compute

$$L_{16} := R_{16} - P_8(\delta_q)R_2$$

in terms of the 40 functions in \mathcal{T}_{16} . Then we multiply by 17, apply $U_{0,17}^*$ and reduce mod 17 to find

$$(6.22) \quad U_{0,17}^*(C_2) \equiv U_{0,17}^*(8E_4\Delta P) \pmod{17}.$$

Next we compute

$$L_{18} := R_{18} - P_9(\delta_q)R_2$$

in terms of the 52 functions in \mathcal{T}_{18} . Then apply $U_{0,17}^*$, reduce mod 17 and use (6.22) to find

$$U_{0,17}^*(R_2) \equiv U_{0,17}^*(8E_4\Delta P + 11E_6\Delta P) \pmod{17}.$$

Thus we have

$$\text{SPT}(17, 5) = \tfrac{1}{2}U_{0,17}^*(C_2 - R_2) \equiv U_{0,17}^*(3E_6\Delta P).$$

By Theorem 3.4

$$U_{0,17}^*(3E_6\Delta P) \equiv E^7(q)G(\tau) \pmod{17},$$

for some $G(\tau) \in M_{22}(1) \cap \mathbb{Z}[[q]]$. A finite computation shows that

$$(6.23) \quad G(\tau) \equiv 14E_{16}(\tau)E_6(\tau) \pmod{17}.$$

In fact we need only verify the coefficients of q^n on both sides of (6.23) agree mod 17 for $n \leq \lfloor \frac{22}{12} \rfloor = 2$. This is a standard argument. See for example [30]. But by [28, Theorem 2(i)]

$$E_{16}(\tau) \equiv 1 \pmod{17},$$

and we have

$$\text{SPT}(17, 5) \equiv 14E^7(q)E_{16}(\tau)E_6(\tau) \equiv 14E^7(q)E_6(\tau) \pmod{17},$$

which is (6.5). Thus

$$\sum_{n=0}^{\infty} \text{spt}(17n + 5)q^{24n+7} \equiv 14G_{17}(\tau) \pmod{17},$$

where

$$G_{17}(\tau) = \eta^7(24\tau)E_6(24\tau) \in S_{\frac{19}{2}}(576, \chi_{12}).$$

Since $\dim M_6(1) = 1$, $G_{17}(\tau)$ is a Hecke eigenform by Corollary 3.2. The first eigenvalue divisible by 17 is

$$\lambda_7 = -24959264 = -2^5 \cdot 11 \cdot 17 \cdot 43 \cdot 97$$

We want the smallest integer c satisfying $(c, 7) = 1$ and $c \equiv (-17)7 \equiv 1 \pmod{24}$; i.e. $c = 1$. As in (6.21) we obtain the congruence

$$\text{spt}(7^4 \cdot 17 \cdot n + 243) \equiv 0 \pmod{17},$$

which is (6.12).

$\ell = 19$. We find

$$\begin{aligned} U_{0,19}^*(C_2) &\equiv U_{0,19}^*(5E_6\Delta P) \pmod{19}, \\ U_{0,19}^*(R_2) &\equiv U_{0,19}^*(5E_6\Delta P + E_4^2\Delta P) \pmod{19}, \\ \text{SPT}(19, 4) &= \frac{1}{2}U_{0,19}^*(C_2 - R_2) \equiv U_{0,19}^*(9E_4^2\Delta P). \end{aligned}$$

Using Theorem 3.4 we find that

$$U_{0,19}^*(9E_4^2\Delta P) \equiv 10E^5(q)E_{18}(\tau)E_4^2(\tau) \equiv 10E^5(q)E_4^2(\tau) \pmod{19}.$$

The result (6.6) follows. Thus

$$\sum_{n=0}^{\infty} \text{spt}(19n+4)q^{24n+5} \equiv 10G_{19}(\tau) \pmod{19},$$

where

$$G_{19}(\tau) = \eta^5(24\tau)E_4^2(24\tau) \in S_{\frac{21}{2}}(576, \chi_{12}).$$

Since $\dim M_8(1) = 1$, $G_{19}(\tau)$ is a Hecke eigenform by Corollary 3.2. The first eigenvalue divisible by 19 is λ_5 and (6.13) follows.

$\ell = 23$. We find

$$\begin{aligned} U_{0,23}^*(C_2) &\equiv U_{0,23}^*(2E_{10}\Delta P) \pmod{23}, \\ U_{0,23}^*(R_2) &\equiv U_{0,23}^*(2E_{10}\Delta P + 9\Delta^2P + 21E_4^3\Delta P) \pmod{23}, \\ \text{SPT}(23, 1) &= \frac{1}{2}U_{0,23}^*(C_2 - R_2) \equiv U_{0,23}^*(7\Delta^2P + E_4^3\Delta P). \end{aligned}$$

By Theorem 3.4

$$U_{0,23}^*(7\Delta^2P + E_4^3\Delta P) \equiv E(q)E_{22}(\tau)(7\Delta(\tau) + E_4^3(\tau)) \equiv E(q)(7\Delta(\tau) + E_4^3(\tau)) \pmod{23}.$$

The result (6.7) follows. However in this case the function $E(q)(7\Delta(\tau) + E_4^3(\tau))$ is not a Hecke eigenform.

$\ell = 29$. We find

$$\begin{aligned} U_{0,29}^*(C_2) &\equiv U_{0,29}^*(\Delta P(2E_4^2 + 20E_4^4 + 11\Delta E_4)) \pmod{29}, \\ U_{0,29}^*(R_2) &\equiv U_{0,29}^*(\Delta P(25E_4^2 + 20E_4^4 + 11\Delta E_4 + 5\Delta E_6)) \pmod{29}, \\ \text{SPT}(29, 23) &= \frac{1}{2}U_{0,29}^*(C_2 - R_2) \equiv U_{0,29}^*(\Delta P(3E_4^2 + 12\Delta E_6)) \pmod{29}. \end{aligned}$$

Using Theorem 3.4 we find $U_{0,29}^*(E_4^2\Delta P) \equiv 0 \pmod{29}$ and

$$U_{0,29}^*(12E_6\Delta^2P) \equiv 17E^{19}(q)E_{28}(\tau)E_6(\tau) \equiv 17E^{19}(q)E_6(\tau) \pmod{29}.$$

The result (6.8) follows. Thus

$$\sum_{n=0}^{\infty} \text{spt}(29n+23)q^{24n+23} \equiv 17G_{29}(\tau) \pmod{29},$$

where

$$G_{29}(\tau) = \eta^{19}(24\tau)E_6(24\tau) \in S_{\frac{31}{2}}(576, \chi_{12}).$$

Since $\dim M_6(1) = 1$, $G_{29}(\tau)$ is a Hecke eigenform by Corollary 3.2. The first eigenvalue divisible by 29 is λ_{13} and (6.14) follows.

$\ell = 31$. We find

$$\begin{aligned} U_{0,31}^*(C_2) &\equiv U_{0,31}^*(2\Delta P(3E_6\Delta + 6E_6^3 + 2E_6 + 5E_{10})) \pmod{31}, \\ U_{0,31}^*(R_2) &\equiv \Delta P(6E_6\Delta + 29\Delta E_4^2 + 12E_6^3 + 2E_6 + 30E_{10}) \pmod{31}, \\ \text{SPT}(31, 22) &= \frac{1}{2}U_{0,31}^*(C_2 - R_2) \equiv U_{0,31}^*(\Delta P(\Delta E_4^2 + E_6 + 21E_{10})) \pmod{31}. \end{aligned}$$

By using Theorem 3.4 we show that $U_{0,31}^*(E_6\Delta P) \equiv U_{0,31}^*(E_{10}\Delta P) \equiv 0 \pmod{31}$, and

$$U_{0,31}^*(E_4^2\Delta^2 P) \equiv 30E^{17}(q)E_{30}(\tau)E_4^2(\tau) \equiv 30E^{17}(q)E_4^2(\tau) \pmod{31}.$$

The result (6.9) follows. Thus

$$\sum_{n=0}^{\infty} \text{spt}(31n + 22)q^{24n+17} \equiv 30G_{31}(\tau) \pmod{31},$$

where

$$G_{31}(\tau) = \eta^{17}(24\tau)E_4^2(24\tau) \in S_{\frac{33}{2}}(576, \chi_{12}).$$

Since $\dim M_8(1) = 1$, $G_{31}(\tau)$ is a Hecke eigenform by Corollary 3.2. The first eigenvalue divisible by 31 is λ_{29} and (6.15) follows.

$\ell = 37$. We find

$$\begin{aligned} U_{0,37}^*(C_2) &\equiv U_{0,37}^*(\Delta P(22\Delta^2 + 4\Delta + 31\Delta E_6^2 + 17\Delta E_4 + 4E_4^3 + 3E_{10} \\ &\quad + 27E_4^4 + 12E_4^6)) \pmod{37}, \\ U_{0,37}^*(R_2) &\equiv U_{0,37}^*(\Delta P(22\Delta^2 + 36\Delta + 31\Delta E_6^2 + 35\Delta E_4 + 20\Delta E_6 E_4^2 + 36E_4^4 \\ &\quad + 13E_{14}E_4^3 + 23E_{10} + 36E_4^3 + 12E_4^6)) \pmod{37}, \\ \text{SPT}(37, 17) &= \frac{1}{2}U_{0,37}^*(C_2 - R_2) \\ &\equiv U_{0,37}^*(\Delta P(21\Delta + 28\Delta E_4 + 27\Delta E_6 E_4^2 + 27E_{10} + 14E_4^4 + 21E_4^3 + 12E_{14}E_4^3)) \pmod{37}. \end{aligned}$$

Using Theorem 3.4 we find that

$$U_{0,37}^*(E_{10}\Delta P) \equiv U_{0,37}^*(\Delta P(\Delta + E_4^3)) \equiv U_{0,37}^*(\Delta P(28\Delta E_4 + 14E_4^4)) \equiv 0 \pmod{37},$$

and

$$U_{0,37}^*(\Delta P(27\Delta E_6 E_4^2 + 12E_{14}E_4^3)) \equiv 12E^{11}(q)E_{36}(\tau)E_4^2(\tau)E_6(\tau) \equiv 12E^{11}(q)E_4^2(\tau)E_6(\tau) \pmod{37}.$$

The result (6.10) follows. Thus

$$\sum_{n=0}^{\infty} \text{spt}(37n + 17)q^{24n+11} \equiv 12G_{37}(\tau) \pmod{37},$$

where

$$G_{37}(\tau) = \eta^{11}(24\tau)E_4^2(24\tau)E_6(24\tau) \in S_{\frac{39}{2}}(576, \chi_{12}).$$

Since $\dim M_{14}(1) = 1$, $G_{37}(\tau)$ is a Hecke eigenform by Corollary 3.2. The first eigenvalue divisible by 37 is λ_5 and (6.16) follows.

7. CONCLUDING REMARKS

In a companion paper [14] we further explore the general problem of congruences mod ℓ for rank moments, Andrews [2] symmetrized rank moments and full rank functions for k -marked Durfee symbols for general ℓ . Let $\ell > 3$ be prime, and β_ℓ, r_ℓ be as in Theorem 3.4. By a detailed ℓ -adic analysis of the rank-crank moment equation (5.9) we have been able to show that

$$(7.1) \quad \sum_{n=0}^{\infty} N_{2k}(\ell n + \beta_\ell) q^{24n+r_\ell} \equiv \eta^{r_\ell}(24\tau) G_{\ell,2k}(24\tau) \pmod{\ell},$$

where $G_{\ell,2k}(\tau)$ is a sum of level one integral modular forms of bounded weight when $2 \leq 2k \leq \ell - 3$. When $2k = \ell - 1$ the result is similar but involves an additional function analogous to Ψ , given in (5.14). By using a result of Treneer [31, Prop. 5.2] it can be shown that there exists a positive proportion of the primes Q for which the appropriate Hecke operator $T(Q^2)$ annihilates mod ℓ all the half integer weight forms that occur on the right side of (7.1). This implies that there exists a positive proportion of the primes Q such that

$$(7.2) \quad N_{2k}(\ell Q^4 n + \mu) \equiv 0 \pmod{\ell},$$

for all integers $k, n \geq 0$. Here μ is an integer constant that depends on ℓ and Q . As a result this leads to infinitely many Ramanujan-type congruences for Dyson's rank function $N(r, \ell, n)$ (the number of partitions of n with rank congruent to $r \pmod{\ell}$). In particular, this implies that for each prime $\ell > 3$ there is a positive proportion of the primes Q such that

$$(7.3) \quad N(r, \ell, \ell Q^4 n + \mu) \equiv 0 \pmod{\ell},$$

for all integers $n \geq 0$ and $0 \leq r \leq \ell - 1$. Again μ is an integer constant at depends on ℓ and Q . This corresponds to Bringmann's [12] result (1.12) (with $m = u = 1$) which was proved using the theory of Maass forms. As noted before, since $\text{spt}(n)$ involves $p(n)$ and the second rank moment $N_2(n)$ this leads to another proof that there are infinitely many Ramanujan-type congruences mod ℓ ($\ell > 3$ prime) for $\text{spt}(n)$. In particular, this implies that for each prime $\ell > 3$ there is a positive proportion of the primes Q such that

$$(7.4) \quad \text{spt}(\ell Q^4 n + \mu) \equiv 0 \pmod{\ell},$$

for all integers $k, n \geq 0$ and μ is an integer constant at depends on ℓ and Q . We note that Bringmann [13] proved this result by a different method.

In a recent paper, Bringmann, Ono and Rhoades [16] have shown the generating function for many rank differences is a modular form. In particular, they have shown the function

$$(7.5) \quad \sum_{n=0}^{\infty} (N(r_1, \ell, \ell n + d) - N(r_2, \ell, \ell n + d)) q^{24(\ell n + d) - 1}$$

is a weakly holomorphic weight $1/2$ modular form when $1 - 24d$ is a quadratic nonresidue mod ℓ and when it is a quadratic residue under certain additional conditions on r_1 and r_2 . As noted in [16] this was observed earlier by Atkin and Swinnerton-Dyer [11] in the cases $\ell = 5$ and $\ell = 7$. It should also be noted that it also been observed when $\ell = 11$ by Atkin and Hussain [9], and $\ell = 13$ by O'Brien [24]. What is more interesting is the result also holds in the case $24d \equiv 1 \pmod{\ell}$ for $\ell = 11$ and $\ell = 13$ (again [9] and [24]). We conjecture that the analogue for the case $24d \equiv 1 \pmod{\ell}$ holds for all primes $\ell > 3$. The cases $\ell = 5, 7$ are trivial since the rank differences are zero. We can prove that in the case $24d \equiv 1 \pmod{\ell}$ the generating function for the rank differences is congruent mod ℓ to a

sum of half-integer weight cusps. An analog of this also holds in the case that $1 - 24d$ is a quadratic nonresidue mod ℓ .

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